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Engineering students' representational flexibility - the case of definite integral

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ABSTRACT: This study examined students' representational flexibility and the learning of the integral concept. By applying the representational framework for examining mathematics education in universities as the research guide, the author analysed and interpreted students' responses to interview questions. In this study, representational flexibility included the ability to work within a representation system, transfer seamlessly between the systems of specific concepts, engage in procedural and conceptual interaction with specific representations and the visual representations students use to resolve specific problems. The analysis of the interviews and results indicated that coordination between the process concept of the graphic representations and the visual ability of the integral problems is necessary for excellent representational flexibility regarding the concept of definite integrals.

INTRODUCTION

Calculus programmes are a crucial subject of university-level mathematics education. Additionally, they are a professional subject and prerequisite knowledge of future workplaces for engineering students. Integrals are a core concept of the calculus curricula and languages, and tools adopted by other fields, such as physics, engineering, economics and statistics. Mathematics education research communities have discussed extensively students' learning and development of mathematical concepts. Topics that have been discussed include the concepts of process and object [1] and different representations [2]. Students' ability to convert process-objects and representations in mathematical concepts require and involve flexible thinking for mathematical concepts. Regarding engineering education, numerous studies have highlighted that mathematics instruction for engineering students should comprise not only mathematics knowledge but also training in mathematical thinking [3]. The results showed that training in mathematical thinking is the most important objective of university mathematics education. Developing the mathematical literacy skills of engineering students during their studies as a type of mathematical literacy required by professionals is also crucial. In this study, this structure as the theoretical framework was adopted to investigate the representational flexibility of engineering students regarding the concept of definite integral.

THEORETICAL FRAMEWORK

Representation is an indispensable tool for presenting mathematical concepts, communicating and considering or thinking. Duval maintained that the process of mathematical thinking required not only the use of representation systems but also cognitive integration of representation systems [4]. Based on Duval's analysis, learning and comprehending mathematics require relatively similar semiotic representations. Duval proposed the following two conversions of semiotic representations: treatments, which referred to the conversion of representations in the same representation system; and conversions, which referred to the conversion of representations of the same objects and concepts in different representation systems. Therefore, the significant aspect of the epistemology and understanding of a mathematical concept derives from employing signs of different representation systems to connect corresponding elements of objects. Considering Duval's reasoning and wording, one can infer that the two crucial dimensions of representational versatility are treatments and conversions, or the ability to perform transformations seamlessly within and between representation systems, that is, so-called representational flexibility.

The essence of the concept of integrals is that the process concept and object concept can be presented by connected but different formats. A number of studies have indicated that the representations used by students to solve an integral problem are related to the meanings they attribute to the concept of integrals. The graphical representation of definite integrals is typically used in calculations that involve areas under a curve, whereas numerical representations are used for Riemann's cumulative addition problems. Solving integrals using common integration techniques demonstrates the need for symbolic representations. Based on representations of mathematical objects and process-objects, the representational structure of the concept of definite integral is show in Table 1. Students' definite integral concept is divided or placed into two dimensions, understanding and representation, and expressed in matrix format to depict the representational flexibility of students in various representation systems.

Table 1: The representational flexibility structure of the definite integral concept.

Levels	Representations		
	Symbolic	Graphical	Numerical
Procedure	Can compute integral values using integral formula	Can only calculate areas using symbolic representations	Cannot use numerical approximation to calculate area
Process	Understands the relationship between the integrand and upper and lower limits of integration	Comprehends the relationship between the area above the X-axis and the integral	Can interpret the limiting process of an rectangular area sums
Object	Can interpret and comprehend that a definite integral is an accumulation function	Can interpret and comprehend the relationship between the area and integral	Can interpret and comprehend the limiting process of Riemann sums

METHODOLOGY

The 25 first-year engineering students who participated in this study were enrolled at a university of technology and had learned the basic rules of integration using primitives, as well as their relationship to the calculation of a number of areas under curves. The instruments used for data collection were a questionnaire containing problems and interviews. The questionnaire comprised seven problems in definite integral (Figure 1). These problems enabled the students' performance regarding the coordination of registers and the level of process/object to be analysed. The results of the questionnaire necessitated further investigation into the versatile mathematics thinking of students. Thus, 25 task-based interviews were conducted.

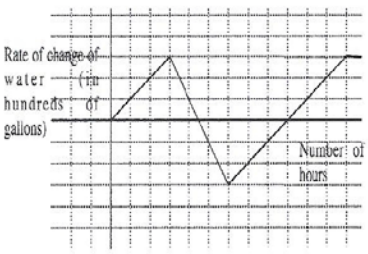
Task 1. If $\int_1^3 f(t)dt = 8.6$, use two strategies to evaluate the value of $\int_2^4 f(t-1)dt$.

Task 2. Let *f* represent the rate at which the amount of water in Phoenix's water tank changed (hundreds of gallons per hour) over a 12-h period from 6 am to 6 pm last Saturday (Assume that the tank was empty at 6 am (t = 0)). Use the graph of f provided below to answer the following questions:

a. How much water was in the tank at noon?

b. What is the meaning of $g(x) = \int_0^x f(t)dt$?

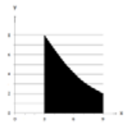
c. What is the value of $g(9)$?



Task 3. Is it true or false that if $\int_a^b f(x)dx \geq \int_a^b g(x)dx$, then $f(x) \geq g(x)$ for all $x \in [a, b]$? Justify your answer.

Task 4. If $\int_1^5 f(x)dx = 10$, use two strategies to evaluate the value of $\int_1^5 (f(x)+2)dx$.

Task 5. Estimate the area of the shaded region.



Task 6. The graph of f is sketched below. Given that $\int_{-2}^{5} f(x)dx = \frac{39}{8}$, determine the value of α.

Task 7. Use two strategies to calculate $\int_{-3}^{3} |x+2|dx$.

Figure 1: The study questionnaire.

To assess and interpret the representational flexibility of the definite integral concept of the engineering students in this study, the author constructed the representational flexibility structure of definite integrals, employed this structure to develop clear standards, and used the triad mechanism proposed by Piaget and Garcia to describe and, then, classify the thinking of the students into various levels [5]. The standards were related to the thinking process adopted and presented by the students when problem solving, as well as their potential to construct relationships among the various representations and properties, and the degree to which they integrated these relationships into their explanation of problem solving. This mechanism divides the representational flexibility of concepts into three stages. At the intra stage, one tends to focus on a single cognitive item, overlooking other actions, processes and objects of a similar nature. At the inter stage, one can perceive and confirm the relationships among various actions, processes and objects. At the trans stage, one can use the relationships identified in the previous stage to construct a consistent structure based on the relationships among various actions, processes and objects.

RESULTS

Students' Representational Flexibility at the Intra Stage

The initial level of representational flexibility when coordinating the construction of developmental chains of process/object and representation is the intra stage. The author categorised nine students into this group. One of the representational flexibility characteristics shared by these students was that they could not recognise the relationship between the area and integral. These students could only process representations within a representation system, and the representations used were influenced by the representation format employed for problems. Additionally, they preferred solving problems using symbolic representations. Consider the following excerpt from the interview conducted with Porter, who has a collection of rules that enable him to integrate fundamental functions, such as the integrals in Tasks 4 and 7. However, Porter could not solve the problems using graphical representations. In Task 3, he stated that the proposition was true and provided specific examples of functions as evidence without giving graphic representations, failing to provide suitable justifications. He provided a specific example that defined two functions $\mathrm{f(x)} = x^2+1$ and $\mathrm{g(x)} = x^2$, and calculated two integrals between 1 and 2 to obtain 10/3 for f and 7/3 for g. Subsequently, the interview progressed as shown below.

R: Can you provide a geometric or numerical example?
P: (Draws the two curves of $\mathrm{f(x)} = x^2+1$ and $\mathrm{g(x)} = x^2$) Like this?
R: Can you do this in graph form without an equation?
P: How do I draw graphs without equations?

Drawing graphs based on the two example functions provided, Porter was unable to think using graphical representations without algebraic formulae. Porter's thinking pattern relied on symbolic but not graphical representation. A similar situation occurred for Task 5. Porter did not use numerical approximation to compute the area; instead, he assumed that the graphical function was a parabola (Figure 2). Subsequently, the interview progressed as shown below.

R: What do you think of this problem?
P: We have to calculate the enclosed area for the parabola, x=3, x=9, and the X axis in this problem.
R: Why is it a parabola?
P: Because the graph looks like a parabola.
R: Is it absolutely necessary to compute the area using the integral?
P: Of course. The integral is used to calculate area.

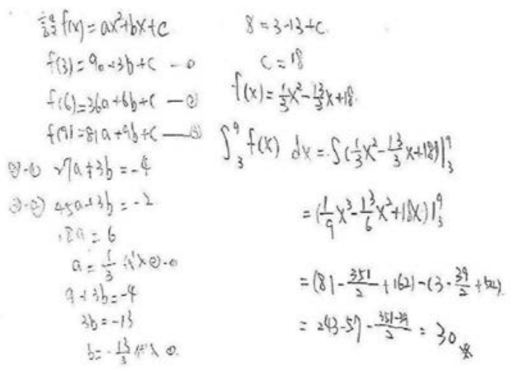


Figure 2: Porter's problem-solving process for Task 5.

To Porter, an integral was simply a tool for computing area. Another student, John, was also categorised into the intra stage. For Task 6, he calculated the linear equation that passed through the two points (-2,-2) and (1/2,8) as 4 x – y + 6 = 0. Then, he calculated the area enclosed by linear lines $\int_{-2}^{1/2}(4\mathrm{x}+6)\mathrm{dx} + \int_{1/2}^{3}\alpha\mathrm{xdx}$, and did not solve the problem using the relationship between the area and integral. In Task 7, John defined the step function and, then, established two integrals (between [–3, 0] and between [0, 3]). When he was asked *Why do you separate the area into these two integrals?*, he answered *I separate the area into two integrals because of absolute values; one integral represents the value to the left of 0, and the other represents the one to the right of 0.*

The students in the intra stage generally used a single representation, and symbolic representation was used to solve all types of problems. This indicates that students consider symbolic representation as a support tool. The high proportion of symbolic representations used in versatile thinking has attracted attention. Additionally, students in this group were inclined to rely on analytical thinking instead of visual thinking. They were incapable of visualising problems. Furthermore, they tend to be cognitively fixed on algorithms and procedures instead of recognising the advantages of visualising the tasks; this is a phenomenon that Eisenberg and Dreyfus described as a reluctance to visualise [6].

Students' Representational Flexibility at the Inter Stage

The next level of representational flexibility of the concept of definite integral regarding the existence of cognitive links and awareness of these links is the inter stage. The author categorised 14 students into this group. These students understood the relationships between representation systems and could change or transfer the representations in some of the representation systems. However, these students had difficulty coordinating these relationships. The students in this group shared the following characteristics: for the numerical representation system, they approached the process concept level; for the graphical representation system, they had reached the process concept level; and for the symbolic representation systems, they had approached or achieved the process concept level. These students could perform treatments and conversions on the three representations for the procedure concept level. Helen was one of the students in this group. She could use correct symbolic representations to perform mathematical thinking and could manipulate the area using graphical representations according to the changes in integral symbols in Tasks 1, 4 and 7. Consider Task 1 for example, Helen assumed that $F'(t) = f(t)$, then $\int_1^3 f(t)dt = F(3) - F(1) = 8.6$. Consequently, $\int_2^4 f(t-1)dt = F(3) - F(1) = 8.6$. Figure 3 shows Helen's graphical representation.

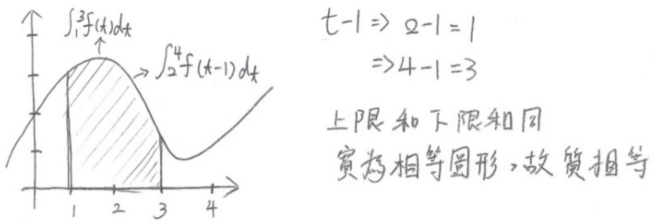


Figure 3: Helen's problem-solving process for Task 3.

However, for Task 3, she said that the proposition was false and gave graphic representations (Figure 4) but failed to make suitable justifications.

R: Can you explain what you think of this task?
H: The area enclosed by f, x = a, x = b, and the x axis is greater than the area enclosed by g, but the function value of f is smaller than g.
R: But the question involves the integral of f being greater than that of g.
H: The integral value is the area; therefore, a greater integral means a greater area.
R: Does this has any relevance to the area being above or below the x axis?
H: It is irrelevant to the area being above or below the x axis.

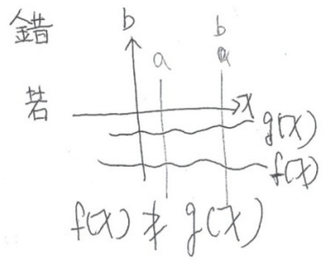


Figure 4: Helen's problem-solving process for Task 3.

Similar to Helen, although the students in this group could convert symbolic representations and graphical representations, they believed that the integral value was the same as the area. Although they understood the relationship between the area above the X axis and the integral, they did not understand the relationship between the area below the x axis and the integral. For example, Mary calculated that the sum of the areas of the two triangles and one trapezoidal in Task 6 (Figure 5) equalled an integral value of 39/8, and that $\alpha = -29/28$; however, she did not realise that the value of α was greater than -1 in the graph.

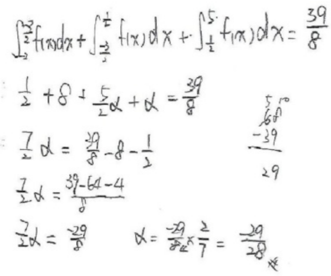


Figure 5: Mary's problem-solving process for Task 6.

In Task 5, Javi divided the region into sections based on the upper and lower limits and, then, calculated the area (Figure 6). He coordinated the given graphic representation with the algebraic representation he established. That the function did not have an algebraic expression did not prevent him solving the problem. However, his selections of the height of the rectangle were limited to the left and right endpoints and the midpoint.

R: Tell me what you think of this problem.
J: No curve equation is provided for this problem; therefore, I segmented the area into six rectangles. The sum of the areas of the six rectangles is an approximate value of the original area.
R: Why did you segment the area into six rectangles?
J: Because I had insufficient time. If I had, I would have segmented the area into 10, 12, or even more rectangles.
R: What difference does it make to segment the area into six or 12 rectangles?
J: The more rectangles I have, the more accurate my answer will be.
R: Do you have other methods that can be used to determine the height of the rectangle aside from the function values of the left and right endpoints?
J: The function value of the midpoint can also be used as the height of the rectangle.
R: Anything else?
J: No, that is all.

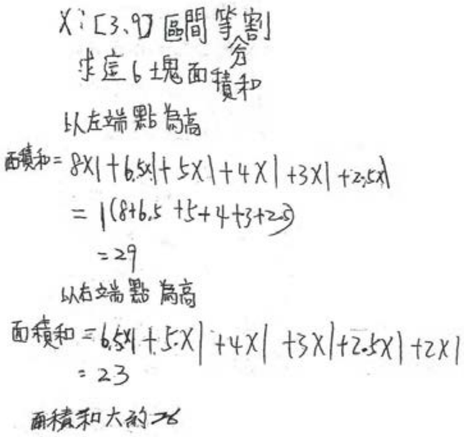


Figure 6: Javi's problem-solving process for Task 6.

These students can perform representation treatments in separate representation systems and generalise, abstract or interiorise these procedures into processes. They can also perform representation conversions in some of the representation systems and interpret the significance of definite integral in various representation systems. Students in this group differ from those at the intra stage in that they have developed visual methods to *see* mathematical concepts and problems better. Although their visual thinking inclines toward local not global thinking, this restricted visualisation actually hinders their solving of the tasks.

Students' Representational Flexibility at the Trans Stage

Two students were categorised into this group. These students could recognise the relationships in representation systems and convert representations between representation systems. The shared characteristics of these students were that their understanding of numerical, graphical, and symbolic representation systems approached the object concept level, and that they could perform treatments and conversions on the three representations at the procedure and process concept levels. In Tasks 1, 4, and 7, Keven used correct symbolic representations to perform mathematical thinking. He also manipulated the area using graphical representations according to the changes in integral symbols. Consider Task 4 for example, Kevin actually employed three methods to solve the problem. The first method was the standard algorithm $\int_1^5 (f(x)+2)dx = \int_1^5 f(x)dx + \int_1^5 2dx = 18$; the second method was the mean value theorem for integrals $f_{ave} = 1/4\int_1^5 f(x)dx = \frac{5}{2}$, $\int_1^5 (f(x)+2)dx = (f_{ave}+2)\times(5-1) = 18$; and the third method was graphical representation (Figure 7).

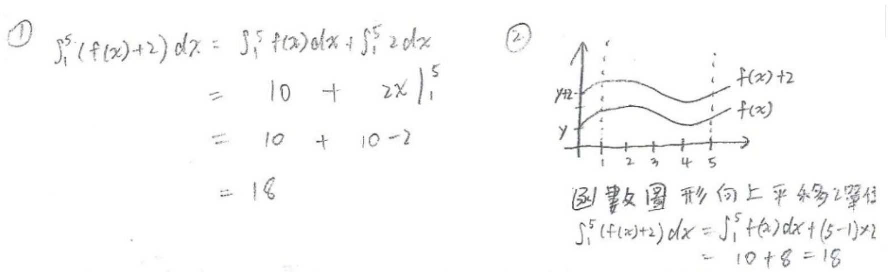


Figure 7: Kevin's problem-solving process for Task 4.

Unlike students in the intra stage who can only apply symbolic representation thinking, Kevin could employ graphical representation as a thinking tool. Additionally, Kevin clearly understood that the area above the x axis was the integral value, and he understood the relationship between the area under the x axis and the integral. As demonstrated by his answer to Task 2, Kevin clearly understood the conversions of numerical, symbolic and graphical representations for the definite integral.

R: How did you calculate the amount of water at noon?
K: I calculated the area from six o'clock in the morning to noon.
R: Why did you calculate the area?
K: Because the Y axis represents the variability of the water inflow and the X axis represents the time elapsed. The product of the two is the volume of the water inflow, which is the area.
R: Why did you subtract 225 from 675?
K: Because 675 is the area above the X axis, which represents the amount of water that flows into the tank. Whereas 225 is the area below the X axis, which represents the amount of water that flows out of the tank. Thus, 675 minus 225 equals the amount of water in the tank at noon.

The most significant difference between the two students in this group and students in the other groups was that the two students in this group had the ability to perform representation treatments in representation systems, and they could perform representation conversions among various representation systems. This ability may involve or be related to visualisation. For Kevin, visualisation is a powerful tool to explore mathematical problems and to ascribe meaning to the concept of definite integrals and the relationship between them.

CONCLUSIONS

In this study, the author recruited first-year engineering students at universities as the research participants to investigate versatile thinking in the concept of definite integrals, using versatile thinking as a theoretical structure. At the intra stage, students memorise or remember a number of integral methods but do not understand the relationship between the area and the definite integral. They can perform representation treatments for individual representation systems, but cannot generalise, abstract or interiorise these procedures into processes. The difficulty these students encounter is the procedural thinking of symbolic representation, not the visual thinking that combines graphical representation.

For example, students at the intra stage require formulae and equations to calculate areas and perform mathematical thinking. The majority of these students were at the inter stage. They had learned the rules for computing definite integrals and had begun recognising various interrelationships. The challenge these students faced was that they tended toward process thinking and not object thinking and their visual thinking patterns inclined toward local thinking and were restricted to specific aspects of the integral concept. Only two students were at the trans stage. These students had learned the rules of calculating derivatives and could recognise interrelationships. They also had visualisation abilities and could coordinate various representations of mathematical schemas, the concept of limit and definite integrals.

The data analysis results show that the main obstacles preventing students from freely shifting within the structure of versatile thinking for the concept of definite integrals were that they had not achieved the process concept for graphical representations of definite integrals, and that this ability involves visualising the abstracted relationships and non-figural information into visual representations and imagery. Based on the students' thinking performance, one can conclude that visual thinking plays a key role in the development of students' versatile thinking. Visualisation is important for versatile thinking because it promotes versatile thinking and encourages students to consider problems holistically before dividing them into parts.

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